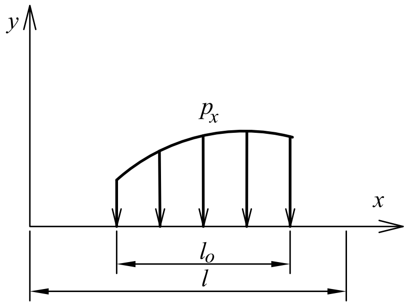
1. RAZVIJANJE FUNKCIJA U TRIGONOMETRIJSKE REDOVE

1.1 Razvijanje funkcija jedne promenljive

Iz mnoštva funkcionalnih redova izdvajaju se trigonometrijski ili Furijeovi redove koji su od fundamentalnog značaja kako za teoriju, tako i za praksu.



Potrebno je funkciju opterećenja p_x definisanu na rasponu l_o izraziti pomoću neke funkcije *p(x)* koja će važiti na celom rasponu *l*. Funkcija *p(x)* može se definisati u obliku beskonačnog reda:

$$p(x) = \sum_{i=0}^{\infty} a_i \varphi_i(x)$$

gde su: $-\varphi_1(x)$ poznate funkcije,

$-a_i$ nepoznati koeficijenti.

Jedan od načina određivanja nepoznatih koeficijenata a_i je da zbir kvadrata grešaka na celom naponu *l* ima minimalnu vrednost, tj.:

$$\min R(a_i) = \min \int_l \left(p_x - p(x)\right)^2 dx$$

$$\frac{\partial R(a_i)}{\partial a_i} = 0 \iff \frac{\partial}{\partial a_i}\left[\int_l \left(p_x - p(x)\right)^2 dx\right] = 0$$

$$\iff \int_l \left(p_x - p(x)\right)\frac{\partial p(x)}{\partial a_i} dx = 0, \quad i = 0,1,2,3,... \qquad (1)$$

Jednačina (1) predstavlja sistem linearnih nehomogenih jednačina iz kojih se mogu odrediti nepoznati koeficijenti a_i .

Račun postaje jednostavniji ako su funkcije $\varphi(x)$ ortogonalne, tj. ako je:

$$\int_l \varphi_n(x)\,\varphi_m(x)\,dx = 0 \quad \text{za} \quad m \neq n$$

$$\int_l \varphi_n(x)\,\varphi_m(x)\,dx = 1 \quad \text{za} \quad m = n.,$$

U tom slučaju u jednačinama (1) pojavljuje se samo po jedan nepoznati koeficijent. U ovakve ortogonalne sisteme spadaju i trigonometrijski redovi, tj. redovi koji sadrže sinusne i kosinusne funkcije.

Periodična funkcija $p(x+l) = p(x)$ može se napisati u sledećem obliku:

$$p(x)=\frac{1}{2}a_o+a_1\cos\frac{2\pi x}{l}+a_2\cos 2\frac{2\pi x}{l}+...+a_n\cos n\frac{2\pi x}{l}+b_1\sin\frac{2\pi x}{l}+b_2\sin 2\frac{2\pi x}{l}+...b_n\sin n\frac{2\pi x}{l}$$

$$p(x)=\frac{a_o}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{l}+b_n\sin\frac{2n\pi x}{l}\right) \tag{2}$$

Radi bolje aproksimacije nije uvek celishodno da se kao perioda izabere samo raspon *l*, već je ponekad pogodnije da se za periodu uvede i višestruka vrednost raspona *l*. Zbog toga je potrebno i opterećenje na podesan način nastaviti preko raspona *l* i dopuniti ga u periodično opterećenje. Zbog toga trigonometrijski red (2) može se napisati u opštem slučaju za neku periodu *L(L=l, 2l, 3l,...)*. Kako će se dalje redovima prikazivati ne samo opterećenje, nego i komponente deformacija i napona, onda će se umesto p_x i *p(x)* pisati kao i u matematici f_x i *f(x)*, pa je

$$f(x)=\frac{a_o}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{L}+b_n\sin\frac{2n\pi x}{L}\right) \tag{3}$$

Određivanje nepoznatih koeficijenata a_o, a_n i b_n

Potrebno je odrediti integral funkcije *f(x)*:

$$\int_x^{x+L} f(x)dx=\frac{a_o}{2}\int_x^{x+L}dx+a_1\int_x^{x+L}\cos\frac{2\pi x}{L}dx+...+a_n\int_x^{x+L}\cos\frac{2n\pi x}{L}dx+...+b_1\int_x^{x+L}\sin\frac{2\pi x}{L}dx$$
$$+...+b_n\int_x^{x+L}\sin\frac{2n\pi x}{L}dx$$
$$\int_x^{x+L} f(x)dx=\frac{1}{2}a_0x\Big|_x^{x+L}=\frac{1}{2}a_oL \quad\Rightarrow\quad a_0=\frac{2}{L}\int_x^{x+L}f(x)\,dx$$

Da bi se odredili koeficijenti $a_n\,(n=1,2,3,...)$, jednačinu (3) treba pomnožiti sa $\cos\frac{2n\pi x}{L}$ i izvršiti integraciju od *x* do *x+L*:

$$a_n=\frac{2}{L}\int_x^{x+L}f_x\cos\frac{2n\pi x}{L}dx$$

Ovde su korišćeni uslovi ortogonalnosti:

$$\int_x^{x+L}\cos\frac{2n\pi x}{L}\cos\frac{2m\pi x}{L}dx=\begin{cases}0,\text{ za } m\neq n\\ \frac{L}{2},\text{ za } m=n\end{cases}$$
$$\int_x^{x+L}\sin\frac{2n\pi x}{L}\cos\frac{2n\pi x}{L}dx=0$$
$$\int_x^{x+L}\sin\frac{2n\pi x}{L}\sin\frac{2m\pi x}{L}dx=\begin{cases}0,\text{ za } m\neq n\\ \frac{L}{2},\text{ za } m=n\end{cases}$$

Slično, za određivanje koeficijenta $b_n (n = 1,2,3,...)$, jednačinu (3) treba pomnožiti sa $\sin\frac{2n\pi x}{L}$ i izvršiti integraciju u granicama od x do $x+L$:

$$b_n = \frac{2}{L}\int_{x}^{x+L} f_x \sin\frac{2n\pi x}{L}dx$$

Ako je funkcija koju treba razviti parna, tj. $f(-x) = f(x)$, onda važi $b_n = 0$, pa je:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L}$$

$$a_n = \frac{2}{L}\left[\int_{-L/2}^{0} f(-x)\cos\frac{2n\pi(-x)}{L}dx + \int_{0}^{L/2} f(x)\cos\frac{2n\pi x}{L}dx\right] = \frac{2}{L}2\int_{0}^{L/2} f(x)\cos\frac{2n\pi x}{L}$$

$$a_n = \frac{4}{L}\int_{0}^{L/2} f(x)\cos\frac{2n\pi x}{L}dx$$

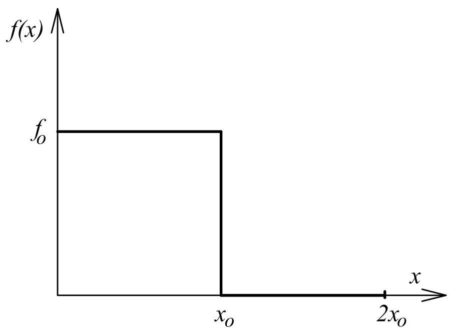
U slučaju neparne funkcije, tj: $f(-x)=-f(x)$ važi $a_n=0$, pa je slično:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{L}$$

$$b_n = \frac{4}{L}\int_{0}^{L/2} f(x)\sin\frac{2n\pi x}{L}dx$$

Primer 1

Datu funkciju razviti u Furijeov red na tri načina:
a) tako da sadrži i sinusne i kosinusne funkcije,
b) kao neparnu funkciju,
c) kao parnu funkciju.



Rešenje:

a) $L=2x_0$

$$a_0 = \frac{2}{L}\int_{0}^{2x_0} f(x)\,dx = \frac{2}{2x_0}\left[\int_{0}^{x_0} f_0 dx + \int_{x_0}^{2x_0} 0\cdot dx\right] = \frac{1}{x_0} = f_0 x\Big|_{0}^{x_0} = f_0$$

$$a_n = \frac{2}{L}\int_{0}^{2x_0} f(x)\cos\frac{2n\pi x}{L}dx = \frac{2f_0}{2x_0}\int_{0}^{x_0}\cos\frac{2n\pi x}{2x_0}dx =$$

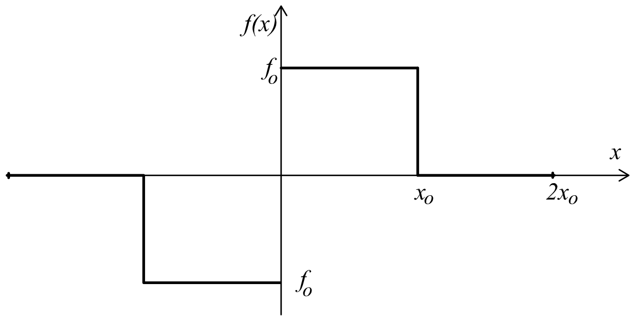
$$= \frac{f_0}{x_0}\frac{x_0}{n\pi}\sin\frac{n\pi x}{x_0}\Bigg|_0^{x_0} = \frac{f_0}{n\pi}(\sin n\pi - 0) = 0$$

$$b_n = \frac{2}{L}\int_0^{2x_0} f(x)\sin\frac{2n\pi x}{L}dx = \frac{2}{2x_0}\int_0^{x_0} f_0 \sin\frac{2n\pi x}{2x_0}dx = \frac{2f_0}{2x_0}\frac{x_0}{n\pi}\left(-\cos\frac{n\pi x}{x_0}\right)\Bigg|_0^{x_0} =$$

$$= -\frac{f_0}{n\pi}(\cos n\pi - 1) = \begin{cases} \frac{2f_0}{n\pi}, & \text{za} \quad n = 1,3,5,... \\ 0, & \text{za} \quad n = 2,4,6,... \end{cases}$$

$$f(x) = \frac{1}{2}f_0 + \frac{2f_0}{\pi}\left(\sin\frac{\pi x}{x_0} + \frac{1}{3}\sin\frac{3\pi x}{x_0} + \frac{1}{5}\sin\frac{5\pi x}{x_0} + ...\right)$$

b) $f(-x) = -f(x)$, $L=4x_0$



$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{L}$$

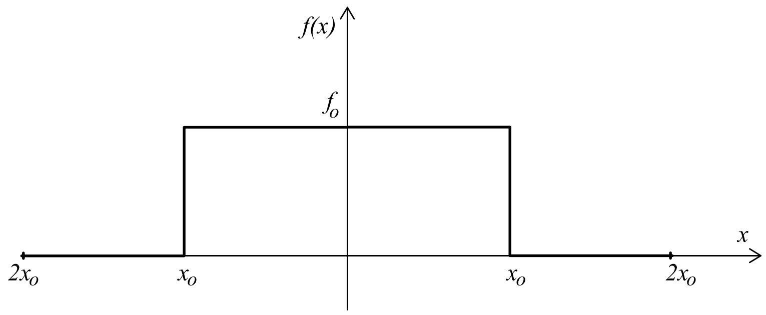
$$b_n = \frac{4}{L}\int_0^{L/2} f_x \sin\frac{2n\pi x}{L}dx = \frac{4}{4x_0}\int_0^{2x_0} f_x \sin\frac{2n\pi x}{4x_0}dx = \frac{1}{x_0}\int_0^{x_0} f_0 \sin\frac{n\pi x_0}{2x_0}dx = \frac{f_0}{x_0}\frac{2x_0}{n\pi}\left(-\cos\frac{n\pi}{2}\right)\Bigg|_0^{x_0}$$

$$b_n = -\frac{2f_0}{n\pi}(\cos\frac{n\pi}{2} - 1) = \begin{cases} \frac{4f_0}{n\pi}, & \text{za} \quad n = 2,6,10,... \\ \frac{2f_0}{n\pi}, & \text{za} \quad n = 1,3,5,... \\ 0, & \text{za} \quad n = 4,8,12,... \end{cases}$$

$$f(x) = \frac{2f_0}{\pi}\sin\frac{\pi x}{2x_0} + \frac{4f_0}{2\pi}\sin\frac{\pi x}{x_0} + \frac{2f_0}{3\pi}\sin\frac{3\pi x}{2x_0} + \frac{2f_0}{5\pi}\sin\frac{5\pi x}{2x_0} + ...$$

$$f(x) = \frac{2f_0}{\pi}\left(\sin\frac{\pi x}{2x_0} + \sin\frac{\pi x}{x_0} + \frac{1}{3}\sin\frac{3\pi x}{2x_0} + \frac{1}{5}\sin\frac{5\pi x}{2x_0} + ...\right)$$

c) *f(x)=f(-x)*



$$f(x) = \frac{1}{2}a_0 + \sum_{1}^{\infty} a_n \cos\frac{2n\pi x}{L}$$

$$a_0 = \frac{4}{L}\int_0^{L/2} f(x)\,dx = \frac{4}{4x_0}\int_0^{2x_0} f(x)\,dx = \frac{1}{x_0}\int_0^{x_0} f_0 dx = \frac{f_0}{x_0}x_0 = f_0$$

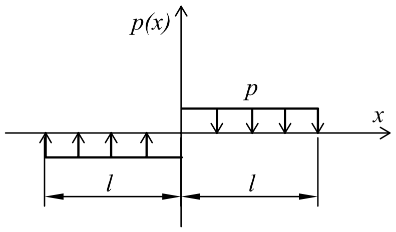
$$a_n = \frac{4}{L}\int_0^{L/2} f(x)\cos\frac{2n\pi x}{L}\,dx = \frac{4}{4x_0}\int_0^{2x_0} f(x)\cos\frac{2n\pi x}{4x_0}\,dx =$$

$$= \frac{1}{x_0}\int_0^{x_0} f_0\cos\frac{n\pi x}{2x_0}\,dx = \frac{f_0}{x_0}\frac{2x_0}{n\pi}\sin\frac{n\pi x}{2x_0}\Bigg|_0^{x_0} = \frac{2f_0}{n\pi}\left(\sin\frac{n\pi}{2} - 0\right) = \frac{2f_0}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} \frac{2f_0}{n\pi}, \text{ za } n = 1,5,9,\ldots \\ 0, \quad \text{ za } n = 2,4,6,\ldots \\ -\frac{2f_0}{n\pi}, \text{ za } n = 3,7,11,\ldots \end{cases}$$

$$f(x) = \frac{f_0}{2} + \frac{2f_0}{\pi}\left(\cos\frac{\pi x}{2x_0} - \frac{1}{3}\cos\frac{3\pi x}{2x_0} + \frac{1}{5}\cos\frac{5\pi x}{2x_0}\ldots\right)$$

Primer 2

Datu funkciju razviti u Furijeov red kao neparnu funkciju: *p(-x)=-p(x).*



L=2l

$$p(x) = \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

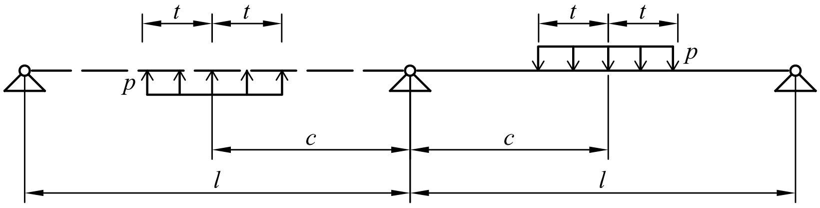
$$b_n = \frac{4}{L}\int_0^{L/2} p_x \sin\frac{2n\pi x}{L}\,dx = \frac{4}{2l}\int_0^{l} p\sin\frac{n\pi x}{l}dx =$$

$$= \frac{2}{l}p\cdot\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)\Bigg|_0^{l} = -\frac{2p}{n\pi}(\cos n\pi - 1) = \begin{cases} \frac{4p}{n\pi}, n = 1,3,5,\ldots \\ 0, n = 2,4,6,\ldots \end{cases}$$

$$p(x)=\frac{4p}{\pi}\sum\frac{1}{n}\sin\frac{n\pi x}{l}=\frac{4p}{\pi}\left(\sin\frac{\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}+...\right)$$

Primer 3

Datu funkciju razviti u Furijeov red kao neparnu funkciju:
p(-x)=-p(x)
L=2l



$$p(x)=\sum_{n=1}^{\infty}\overline{p}_n\sin\frac{2n\pi x}{2l}=\sum_{n=1}^{\infty}\overline{p}_n\sin\frac{n\pi x}{l}$$

$$p(x)=\begin{cases}0\,,0\le x\le c-t\\ \\ p\,,c-t\le x\le c+t\\ \\ 0\,,c+t\le x\le l\end{cases}$$

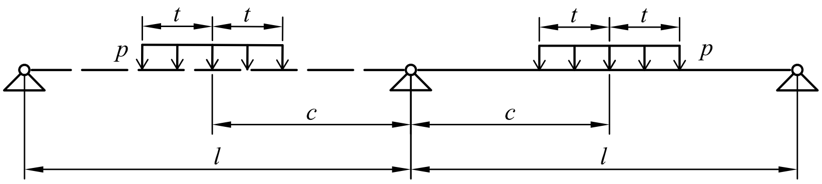
$$\overline{p}_n=\frac{4}{L}\int_0^{L/2}p_x\sin\frac{n\pi x}{l}dx=\frac{4}{2l}\int_0^{l}p_x\sin\frac{n\pi x}{l}dx=\frac{2}{l}p\int_{c-t}^{c+t}\sin\frac{n\pi x}{l}dx=-\frac{2p}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l}\Bigg|_{c-t}^{c+t}=$$

$$=-\frac{2p}{n\pi}\left[\cos\frac{n\pi}{l}(c+t)-\cos\frac{n\pi}{l}(c-t)\right]=-\frac{2p}{n\pi}\cdot(-2)\sin\frac{n\pi c}{l}\sin\frac{n\pi t}{l}=\frac{4p}{n\pi}\sin\frac{n\pi c}{l}\sin\frac{n\pi t}{l}$$

$$p(x)=\frac{4p}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi c}{l}\sin\frac{n\pi t}{l}\sin\frac{n\pi x}{l}$$

Primer 4

Funkciju iz primera 2 razviti u Furijeov red kao parnu.



L=2l,   p(x)=p(-x)

$$p(x)=\frac{1}{2}p_0+\sum_{n=1}^{\infty}p_n\cos\frac{2n\pi x}{2l}=\frac{1}{2}p_0+\sum_{n=1}^{\infty}p_n\cos\frac{n\pi x}{l}$$

$$p_0 = \frac{4}{L}\int_0^{L/2} p(x)dx = \frac{4}{2l}\int_{c-t}^{c+t} pdx = \frac{2p}{l}(c+t-c+t) = \frac{4pt}{l}$$

$$p_n = \frac{4}{L}\int_0^{L/2} p(x)\cos\frac{n\pi x}{l}dx = \frac{4}{2l}\int_{c-t}^{c+t} p\cos\frac{n\pi x}{l}dx =$$

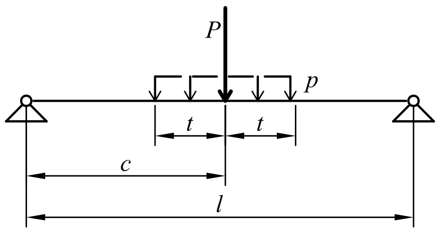
$$= \frac{2p}{l}\frac{l}{n\pi}\sin\frac{n\pi x}{l}\Bigg|_{c-t}^{c+t} = \frac{2p}{n\pi}\left[\sin\frac{n\pi}{l}(c+t) - \sin\frac{n\pi}{l}(c-t)\right] =$$

$$= \frac{2p}{n\pi}2\sin\frac{n\pi t}{l}\cos\frac{n\pi c}{l} = \frac{4p}{n\pi}\sin\frac{n\pi t}{l}\cos\frac{n\pi c}{l}$$

$$p(x) = \frac{2pt}{l} + \frac{4p}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi t}{l}\cos\frac{n\pi c}{l}\cos\frac{n\pi x}{l}$$

Primer 5

Datu funkciju opterećenja razviti u Furijeov red kao neparnu.



$$p = \frac{P}{2t} \Rightarrow P = 2pt$$

Iz prethodnog primera opterećenje p dato u vidu Furijeovog reda glasi:

$$p(x) = \frac{4p}{\pi}\sum_n \frac{1}{n}\sin\frac{n\pi c}{l}\sin\frac{n\pi t}{l}\sin\frac{n\pi x}{l}$$

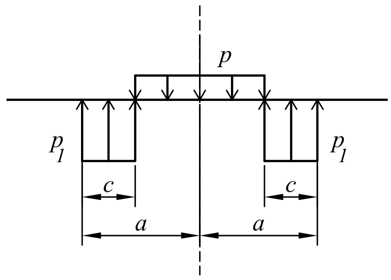
$$p(x) = \lim_{t\to 0}\left[\frac{4p}{\pi}\sum_n \frac{1}{n}\sin\frac{n\pi c}{l}\frac{\sin\frac{n\pi t}{l}}{\frac{n\pi t}{l}}\cdot\frac{n\pi t}{l}\sin\frac{n\pi x}{l}\right]$$

$$p(x) = \frac{2P}{l}\sum_n \sin\frac{n\pi c}{l}\sin\frac{n\pi x}{l}$$

Ovakav red nije kovergentan, ali se i sa tom funkcijom opterećenja ipak dobijaju konvergentni redovi za komponente napona i pomeranja u tačkama nosača koje leže van napadne tačke koncentrisane sile.

Primer 6

Ivično opterećenje dato na slici razviti u Furijeov red kao parnu funkciju za:
a) *L=2a* ; b) *L=4a.*

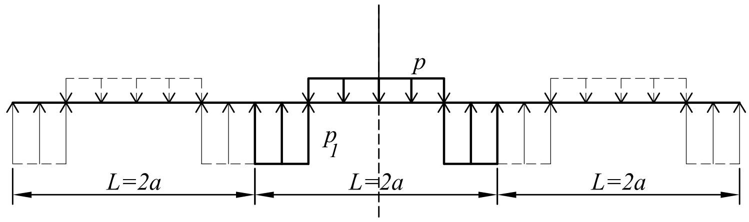


Jednako podeljeno opterećenje p i p_1 čine ravnotežni sistem, pa je:

$$p\cdot 2(a-c)=2p_1\cdot c \quad \Rightarrow p_1=p\cdot\frac{a-c}{c}$$

Kako god da se izabere perioda L, član reda $\frac{1}{2}a_0$ biće jednak nuli, jer je $\int\limits_{-a}^{a} p_x dx=0$

a) $L=2a$:



$$p(x)=\sum_{n=1}^{\infty} a_n\cos\frac{2n\pi x}{L}$$

$$a_n=\frac{4}{L}\int\limits_0^{L/2} p_x\cos\frac{2n\pi x}{L}dx=\frac{4}{2a}\int\limits_0^{a} p_x\cos\frac{n\pi x}{a}dx=\frac{2}{a}\left[\int\limits_0^{a-c} p\cos\frac{n\pi x}{a}dx-\int\limits_{a-c}^{a} p_1\cos\frac{n\pi x}{a}dx\right]=$$

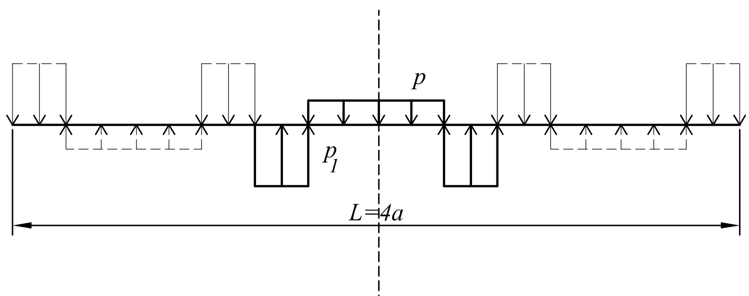
$$=\frac{2}{a}\left[p\frac{a}{n\pi}\sin\frac{n\pi x}{a}\bigg|_0^{a-c}-p\cdot\frac{a-c}{c}\cdot\frac{a}{n\pi}\sin\frac{n\pi x}{a}\bigg|_{a-c}^{a}\right]=\frac{2p}{n\pi}\left[\sin\frac{n\pi}{a}(a-c)+\frac{a-c}{c}\sin\frac{n\pi}{a}(a-c)\right]=$$

$$=\frac{2p}{n\pi}\left(1+\frac{a-c}{c}\right)\sin\frac{n\pi}{a}(a-c)=\frac{2pa}{n\pi c}\sin\frac{n\pi}{a}(a-c)=\frac{2pa}{n\pi c}\left(\sin n\pi\cos\frac{n\pi c}{a}-\sin\frac{n\pi c}{a}\cos n\pi\right)=$$

$$=-\frac{2pa}{n\pi c}\sin\frac{n\pi c}{a}\cos n\pi=-\frac{2pa}{n\pi c}(-1)^n\sin\frac{n\pi c}{a}$$

$$p(x)=-\frac{2pa}{\pi c}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi c}{a}\cos\frac{n\pi x}{a}$$

b) $L=4a$:



$$a_n = \frac{4}{4a}\int_0^{2a} p_x \cos\frac{2n\pi}{4a}dx = \frac{1}{a}\left[\int_0^{a-c} p\cos\frac{n\pi x}{2a}dx - \frac{a-c}{c}\int_{a-c}^{a} p\cos\frac{n\pi x}{2a}dx + \right.$$

$$\left. + \frac{a-c}{c}\int_a^{a+c} p\cos\frac{n\pi x}{2a}dx - \int_{a+c}^{2a} p\cos\frac{n\pi x}{2a}dx \ \right] = \frac{p}{a}\left[\frac{2a}{n\pi}\sin\frac{n\pi x}{2a}\Big|_0^{a-c} - \frac{a-c}{c}\cdot\frac{2a}{n\pi}\sin\frac{n\pi x}{2a}\Big|_{a-c}^{a} + \right.$$

$$+ \frac{a-c}{c}\frac{2a}{n\pi}\sin\frac{n\pi x}{2a}\Big|_a^{a+c} - \frac{2a}{n\pi}\sin\frac{n\pi x}{2a}\Big|_{a+c}^{2a} \ \Big] = \frac{2p}{n\pi}\Big[\sin\frac{n\pi}{2a}(a-c) - \frac{a-c}{c}\sin\frac{n\pi}{2} + \frac{a-c}{c}\sin\frac{n\pi}{2a}(a-c)$$

$$+ \frac{a-c}{c}\sin\frac{n\pi}{2a}(a+c) - \frac{a-c}{c}\sin\frac{n\pi}{2} - \sin n\pi + \sin\frac{n\pi}{2a}(a+c) \ \Big] =$$

$$= \frac{2pa}{n\pi c}\left[\sin\frac{n\pi}{2a}(a-c) + \sin\frac{n\pi}{2a}(a+c) - 2\frac{a-c}{a}\sin\frac{n\pi}{2} \ \right] =$$

$$= \frac{2pa}{n\pi c}\left[2\sin\frac{n\pi}{2}\cos\frac{n\pi c}{a} - 2\frac{a-c}{a}\sin\frac{n\pi}{2}\right] = \frac{4pa}{n\pi c}\sin\frac{n\pi}{2}\left(\cos\frac{n\pi c}{a} - \frac{a-c}{a}\right)$$

$$p(x) = \frac{4pa}{\pi c}\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{n\pi}{2}\left(\cos\frac{n\pi c}{a} - \frac{a-c}{a}\right)\cos\frac{n\pi x}{2a}$$